IRSTI 30.19.21 **DOI:** https://doi.org/10.32523/2220-685X-2025-76-1-7-18
Scientific article

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Method for reconstructing the distribution of unknown spatio-temporal loads in a structure based on viscoelasticity in the Euler-Bernoulli beam coupling

Abstract. In this paper, a novel mathematical model and new approach is proposed for the inverse source problem of recovering the unknown spatial-temporal load $F(x,t)$ in the simply supported non-homogeneous Euler-Bernoulli beam governed by the equation $p(x)\, u_{tt} + \mu(x)\, u_t + (r(x)\, u_{xx})_{xx} + k_L u = F(x,t)$, $(x,t) \in (0,\ell) \times (0, T)$, resting on a viscoelastic foundation, is studied. It is assumed that the rotation at the left boundary $\theta(t) := u_x(0, t)$, $t \in (0, T)$, and also the deflection $u_T(t) := u(x, T)$, $x \in (0, \ell)$ at the final time $T > 0$, are given as measured outputs. The Tikhonov functional $J(F) := (1/2)||\theta - u_x(0,\cdot)||^2_{L^2(0,T)} + (1/2)||u_T - u(\cdot, T)||^2_{L^2(0,l)}$ is introduce to reformulate the inverse problem as a minimization problem for the Tikhonov functional. An explicit gradient formula for this functional is derived. Based on this formula a conjugate gradient algorithm is developed for the considered inverse problem. This algorithm allows to recover the unknown spatial-temporal load with high accuracy, from noise free as well as from random noisy measured outputs.

Keywords: Euler-Bernoulli beam, inverse coefficient problem, Neumann-to-Neumann operator, existence of a quasi-solution, Fr´echet gradient.

Introduction

In real life, the interaction of beams with the soil beneath them changes their behavior. It indicates that the contact has a significant impact on how the beams behave. Therefore, for better structure design, a model of the soil–foundation–structure interaction

Received: 15.01.2025; Revised: 24.02.2025; Approved: 02.03.2025; Available online: 30.03.2025

system that is reasonably realistic is required. One popular soil ideal ization model is the ”linear elastic vertical springs” model, defined as the Winkler foundation and formulated in 1867 by Winkler [1]. The stiffness of the vertical spring is the single parameter in the Winkler model that describes the soil's char acteristics. Despite being the most basic type of elastic foundation, the model is utilized to simulate soil behavior in the majority of real-world applications.

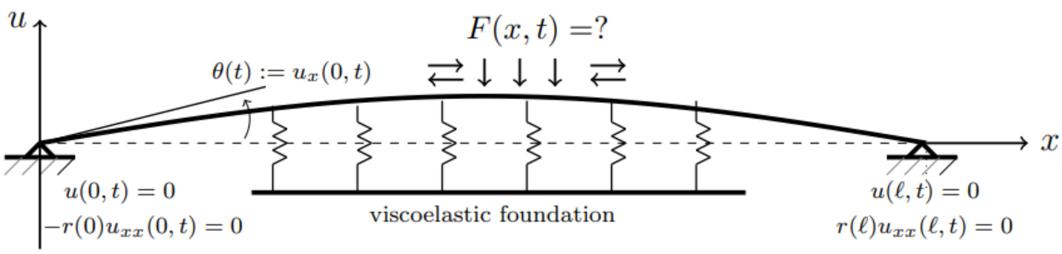

Figure 1 - Simply supported Euler-Bernoulli beam
subjected to interface forces with measured boundary slopes

In this paper, we study the inverse problem of determining the unknown spatial-temporal load $F(x, t)$ in

$$\begin{cases} \rho_A(x)u_{tt} + \mu(x)u_t + (r(x)u_{xx})_{xx} + k_w u = F(x,t), \ (x,t) \in \Omega_T, \\ u(x,0) = 0, \ u_t(x,0) = 0, \quad x \in (0,\ell), \\ u(0,t) = u_{xx}(0,t) = 0, \ u(\ell,t) = u_{xx}(\ell,t) = 0, \ t \in (0,T). \end{cases} \tag{1}$$

from the following boundary and final time measured outputs

$$\begin{cases} \theta(t) := u_x(0,t), \ t \in [0,T], \\ u_T(x) := u(x,T), \ t \in (0,\ell), \end{cases} \tag{2}$$

respectively.

Here, $r(x) = E(x)I(x) > 0$ is the variable flexural rigidity and $\rho_A(x) = \rho(x)A(x)$, while $E(x) > 0$, $I(x) > 0$, $\rho(x)$ and $A(x)$ are the elasticity modulus, the moment of inertia, the mass density and the cross-sectional area, respectively. Further, $\mu(x) \geq 0$ is the damping coefficient, $k_L > 0$ is the stiffness of Winkler foundation. $\Omega_T = \{(x, t) \in \mathbb{R}^2 : 0 < x < \ell, 0 < t < T\}$, and the final time $T > 0$ is finite and may be small enough. The geometry of the inverse problem (1)-(2) is given in Fig. 1.

The study of the dynamic response of beams on viscoelastic foundations subjected to moving loads has been of great significance in engineering. Ghadiri et al. [**1**] studied analytical solution for the steady-state response of an Euler-Bernoulli nanobeam subjected to moving concentrated load and resting on a viscoelastic foundation. For an infinite beam on an elastic foundation, Zheng et. al [4] gave a general solution of a dynamical problem. The effect of the foundation stiffness, traveling speed and length of the beam on the dynamic magnification factor have been studied by Thambiratnam and Zhuge [5] by using the finite element method. Zheng et. al [6] carried out a dynamic analysis for coupled vehicle-bridge vibration system on nonlinear foundation using the Galerkin truncation method.

The importance of the proposed model governed by (1)-(2) is that, it is a generalization of existing mathematical models in the sense that, the Euler-Bernoulli equation (1) includes all the main physical coefficients. Moreover, the load $F(x, t)$ to be determined depends on both the spatial and time variables.In all of the above-mentioned studies, special cases of the load distribution been discussed. Therefore, the algorithm proposed in this study is valid for all of these special cases.

Here, we use the approach introduced in [3] and developed in [**?**]. We derive an explicit gradient formula for the Fr´echet derivative of both components $J_1(F)$ and $J_2(F)$ of the Tikhonov functional $J(F) := J1(F) + J2(F)$ defined as

$$J(F) = 1/2 \parallel \Phi - u_x(0, \cdot; F) \parallel L2(0,T) + 1/2 \parallel \Psi - u(\cdot, T; F) \parallel_{L^2(0,\ell)}, \quad (3)$$

where $u(x, t; F)$ is the solution of the direct problem (1) corresponding to a given $F \in F$ from the set of admissible loads $F \in L^2(\Omega_T)$. This important tool is then employed in solving of concrete inverse problems.

Materials and methods

We assume that the inputs in (1) satisfy the following basic conditions:

$$\begin{cases} \rho_A, r, \mu, \in L^\infty(0,\ell),\ k_w > 0,\ F \in L^2(\Omega), \\ 0 < \rho_0 \le \rho_A(x) \le \rho_1,\ 0 < r_0 \le r(x) \le r_1,\ 0 \le \mu_0 \le \mu(x) \le \mu_1,\ x \in (0,\ell). \end{cases} \tag{4}$$

We define the *set of admissible spatial-temporal loads* $F = \{F \in L^2(\Omega) : \|F\|_{L^2(\Omega)} \le C_F\}$, $C_F > 0$. Introduce the *input-output operators* associated with the inverse problem (1)-(2) as follows:

$$\begin{aligned} &(\Phi F)(t) := u_x(0,t;F),\ t \in [0,T], \\ &(\Psi F)(x) := u(x,T;F),\ x \in (0,\ell),\ F \in \mathcal{F}. \end{aligned} \tag{5}$$

In view of these operators, the inverse problem (1)-(2) can be reformulated as the following system of linear operator equations:

$$\begin{aligned} &(\Phi F)(t) = \theta(t),\ t \in [0,T], \\ &(\Psi F)(x) := u_T(x),\ x \in (0,\ell),\ F \in \mathcal{F}. \end{aligned} \tag{6}$$

However, due to measurement errors in the outputs $\theta(t)$ and $u_T(x)$ exact equality in (6) can not be satisfied. Hence we needs to introduce the Tikhonov functional (3) and reformulate the inverse coefficient problem (1)-(2) as the minimization problem

$$J(F) = \inf_{\tilde{F} \in \mathcal{F}} J(\tilde{F}). \tag{7}$$

Lemma 1 *Assume that the inputs in (1) satisfy the basic conditions (4). Then the Tikhonov functional is Lipschitz continuous, that is*

$$|\mathcal{J}(F_1) - \mathcal{J}(F_2)| \le L_{\mathcal{J}} \|F_1 - F_2\|_{L^2(\Omega_T)},\ F_1, F_2 \in \mathcal{F}, \tag{8}$$

where

$$L_{\mathcal{J}} = \sqrt{3\ell}\, C_1 \left[\sqrt{3\ell}\, C_1 C_F + \|\theta\|_{L^2(0,T)} \right] + \sqrt{T}\, C_2 \left[\sqrt{T}\, C_2 C_F + \|u_T\|_{L^2(0,\ell)} \right] \qquad (9)$$

is the Lipschitz constant, while

$$C_1^2 = \exp(T/\rho_0) - 1, \;\; C_2^2 = \rho_0 \exp(T/\rho_0)/r_0 \qquad (10)$$

and $\rho 0$, $r0 > 0$ *are the constants introduced in (4).*

Proof. We use estimates

$$\|u_t\|_{L^2(0,T;\, L^2(0,\ell))} \le C_1 \|F\|_{L^2(\Omega T)},$$
$$\|u_{xx}\|_{L^2(0,T;\, L^2(0,\ell))} \le C_2 \|F\|_{L^2(\Omega T)},$$

for the weak solution $u \in L^2(0, T; V^2(0, \ell))$, of the initial boundary value problem (1), where $V^2(0, \ell)) := \{v \in H^2(0, \ell): v(0) = v(\ell) = 0\}$, derived in [2] to evaluate the outputs $u_x(0, t)$ and $u(x, T)$. Here $C_1, C_2 > 0$ are the constants introduced in (10). In view of the inequalities

$$\|u_x(0,\cdot)\|^2_{L^2(0,T)} \le 3\ell\, \|u_{xx}\|^2_{L^2(0,T;L^2(0,\ell))},$$
$$\|u(\cdot,T)\|^2_{L^2(0,\ell)} \le T\, \|u_{xx}\|^2_{L^2(0,T;L^2(0,\ell))}$$

these estimates yield:

$$\|u_x(0,\cdot)\|_{L^2(0,T)} \le \sqrt{3\ell}\, C_1 \|F\|_{L^2(\Omega_T)},$$
$$\|u(\cdot,T)\|_{L^2(0,\ell)} \le \sqrt{T}\, C_2 \|F\|_{L^2(\Omega_T)}. \qquad (11)$$

On the other hand, using Corollary 10.1.6 in [3], we can prove that

$$
\begin{aligned}
&|\mathcal{J}(F_1) - \mathcal{J}(F_2)| \\
&\quad \le \frac{1}{2}\left[\|\Phi F_1\|_{L^2(0,T)} + \|\Phi F_2\|_{L^2(0,T)} + 2\|\theta\|_{L^2(0,T)}\right]\|\Phi F_1 - \Phi F_2\|_{L^2(0,T)} \\
&+\frac{1}{2}\left[\|\Psi F_1\|_{L^2(0,T)} + \|\Psi F_2\|_{L^2(0,T)} + 2\|u_T\|_{L^2(0,\ell)}\right]\|\Psi F_1 - \Psi F_2\|_{L^2(0,T)},
\end{aligned} \tag{12}
$$

for all F_1, $F_2 \in F$. By the above definitions of the input-output operators we have:

$$
\begin{aligned}
\|\Phi F_1 - \Phi F_2\|_{L^2(0,T)} &= \|\delta u_x(0,\cdot)\|_{L^2(0,T)}, \\
\|\Psi F_1 - \Psi F_2\|_{L^2(0,T)} &= \|\delta u(\cdot,T)\|_{L^2(0,\ell)},
\end{aligned}
$$

where $\delta u\,(x, t)$ is the solution of problem (1) with the input $\delta F\,(x, t) = F_1(x, t) - F_2\,(x, t)$. Then estimates (11) for this solution are

$$
\begin{aligned}
\|\delta u_x(0,\cdot)\|_{L^2(0,T)} &\le \sqrt{3\ell}\, C_1 \|\delta F\|_{L^2(\Omega_T)}, \\
\|\delta u(\cdot,T)\|_{L^2(0,\ell)} &\le \sqrt{T}\, C_2 \|\delta F\|_{L^2(\Omega_T)}.
\end{aligned}
$$

Using estimates (11) and (13) in (12), and taking into account that $\|F\|_{L^2(\Omega)} \le C_F$ for all $F \in \mathcal{F}$, we deduce that

$$
\begin{aligned}
&|\mathcal{J}(F_1) - \mathcal{J}(F_2)| \\
&\quad \le \sqrt{3\ell}\, C_1 \left[\sqrt{3\ell}\, C_1 C_F + \|\theta\|_{L^2(0,T)}\right]\|\delta F\|_{L^2(0,T)} \\
&\quad +\sqrt{T}\, C_2 \left[\sqrt{T}\, C_2 C_F + \|u_T\|_{L^2(0,\ell)}\right]\|\delta F\|_{L^2(0,T)}.
\end{aligned} \tag{13}
$$

This implies the desired result (8) with the Lipschitz constant defined in (9).

Theorem 1 *Assume that conditions of Lemma 1 hold. Then the minimization problem (8) has a solution in the set of admissible loads* $\mathcal{F} \subset L^2\,(\Omega)$.

Proof. Evidently, the set of admissible loads $\mathcal{F}$ is a nonempty closed convex set in $L^2(\Omega)$. Then, by Theorem 10.1.11, the Lipschitz continuity of the Tikhonov functional implies existence of a solution of the minimization problem (8).

Theorem 2 implies that there exists at least one quasi-solution of the inverse problem (1)-(2).

Denote by $\delta\mathcal{J}(F) := \mathcal{J}(F + \delta F) - \mathcal{J}(F)$, F, $F + \delta F \in \mathcal{F}$ the increment of the Tikhonov functional. Then

$$\begin{aligned}\delta\mathcal{J}(F) &:= \delta\mathcal{J}_1(F) + \delta\mathcal{J}_2(F)\\ &\int_0^T [u_x(0,t;F) - \theta(t)]\,\delta u_x(0,t)dt + \frac{1}{2}\int_0^\ell (\delta u_x(0,t))^2\,dt\\ &\int_0^\ell [u(x,T;g) - u_T(x)]\,\delta u(x,T)dx + \frac{1}{2}\int_0^\ell (\delta u(x,T))^2\,dx.\end{aligned} \tag{14}$$

Multiply now both sides of equation (1) for $\delta u(x, t)$ by arbitrary function $w \in L^2(0, T; V^2(0, \ell))$, integrate over $(0, T)$ and apply the integration by parts formula multiple times. Then we obtain following integral identity:

$$\begin{aligned}&\int_0^T\int_0^\ell [\rho_A(x)w_{tt} - \mu(x)w_t + (r(x)w_{xx})_{xx} + k_L w]\,\delta u\,dxdt\\ &+\int_0^T \left[(r(x)\delta u_{xx})_x w - r(x)\delta u_{xx}w_x + r(x)\delta u_x w_{xx} - \delta u\,(r(x)w_{xx})_x\right]_{x=0}^{x=\ell} dt\\ &\qquad+\int_0^\ell [\rho_A(x)\delta u_t w - \rho_A(x)\delta u w_t + \mu(x)\delta u w]_{t=0}^{t=T}\,dx\\ &\qquad\qquad= \int_0^T\int_0^\ell \delta F(x,t)w\,dxdt.\end{aligned} \tag{15}$$

First, we require that the arbitrary function $w(x, t)$ is the solution of the following backward problem with the Neumann input $u_x(0, t; F) - \theta(t)$:

$$\begin{cases} \rho_A(x)\phi_{tt} - \mu(x)\phi_t + (r(x)\phi_{xx})_{xx} + k_L\phi = 0, \ (x,t) \in \Omega_T, \\ \phi(x,T) = 0, \ \phi_t(x,T) = 0, \ x \in (0,\ell), \\ \phi(0,t) = 0, \ -r(0)\phi_{xx}(0,t) = u_x(0,t;F) - \theta(t), \\ \phi(\ell,t) = \phi_{xx}(\ell,t) = 0, \ t \in [0,T]. \end{cases} \tag{16}$$

Then we obtain from the integral identity (15) the following input-output relationship:

$$\int_0^T [u_x(0,t;F) - \theta(t)]\, \delta u(0,t)\, dt = \int_0^T \int_0^\ell \delta F(x,t)\phi(x,t)\, dxdt. \tag{17}$$

Next we require that the arbitrary function w(x, t) is the solution of the backward problem with the final time input $-[u\,(x, T; F) - u_T\,(x)] \,/\, \rho_A\,(x)$:

$$\begin{cases} \rho_A(x)\psi_{tt} - \mu(x)\psi_t + (r(x)\psi_{xx})_{xx} + k_L\psi = 0, \ (x,t) \in \Omega_T, \\ \psi(x,T) = 0, \ \psi_t(x,T) = \dfrac{1}{\rho_A(x)} \left[u(x,T) - u_T(x)\right], \ x \in (0,\ell), \\ \phi(0,t) = \phi_{xx}(0,t) = \phi(\ell,t) = \phi_{xx}(\ell,t) = 0, \ t \in [0,T]. \end{cases} \tag{18}$$

Then the integral identity (15) implies the following *input-output relationship:*

$$\int_0^\ell [u(x,T) - u_T(x)]\, \delta u(x,T)\, dx = \int_0^T \int_0^\ell \delta F(x,t)\psi(x,t)\, dxdt. \tag{19}$$

Theorem 2 Assume that in addition to the basic conditions (4), the inputs in (1) and also the measured output θ (t) satisfy the following regularity conditions:

$$r \in H^2(0,\ell), \ F_t \in L^2(\Omega_T), \ \theta \in H^1(0,T). \tag{20}$$

Then the Tikhonov functional is Fr´echet differentiable. Furthermore, for its Fr´echet gradient the following formula holds:

$$\nabla \mathcal{J}(F)(x,t) = \phi(x,t) + \psi(x,t),\ (x,t) \in \Omega_T,\ F \in \mathcal{F}, \tag{21}$$

where φ(x, t) and ψ(x, t) are the weak solutions of the adjoint problems (16) and (18), respectively.

Proof. Evidently, the set of admissible loads F is a nonempty closed convex set in $L^2(\Omega)$. Then, by Theorem 10.1.11, the Lipschitz continuity of the Tikhonov functional implies existence of a solution of the minimization problem (8).

This implies the desired result (8) with the Lipschitz constant defined in (9).

Discussion

The vibrations and stability of uniform beams resting on continuous twoparameter elastic foundation were studied. The equation of motion for Timoshenko and Bernoulli- Euler beam was derived. The relationships between the parameters describing vibration, the compressive force and the foundation parameters were investigated. The methodology of mathematical modelling developed for the simply supported non-homogeneous Euler-Bernoulli beam on a Winkler foundation allows us to identify the relationship between the load and the material characteristics. The individual effect of foundation stiffness parameters, transverse shear deformation and rotatory inertia on eigenvalues of the beam can be examined by performing a parametric study.

Conclusion

In this study, we propose a new model of a source identification problem for a supported Euler-Bernoulli beam on a viscoelastic foundation, based on boundary and final time measurements. Based on analysis of the input-output operators, we derive a gradient formula for recovering the unknown spatial-temporal load distribution. Numerical experiments performed for real data problems show high accuracy of the proposed algorithm. According to presented results.

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Эйлер-Бернулли арқалық муфтасында тұтқыр серпімділікке негізделген құрылымдағы белгісіз кеңістіктік-уақыттық жүктемелердің таралуын қайта құру әдісі

Аңдатпа. Бұл жұмыста $p(x)\ u_{tt} + \mu(x)u_t + (r(x)\ u_{xx})_{xx} + k_L u = F(x,t),\ (x,t) \in (0,\ell) \times (0,\ T)$, теңдеуімен басқарылатын жай ғана қолдайтын біртекті емес Эйлер-Бернулли шоғырында белгісіз кеңістіктік-уақыттық жүктемені $F(x,t)$ қалпына келтірудің кері көзі мәселесіне жаңа математикалық модель және жаңа тәсіл ұсынылған. t) ∈ (0, ℓ) ×(0, T), тұтқыр серпімді іргетасқа тірелген, зерттеледі. Сол жақ шекарадағы $\theta\ (t) := u_x(0,\ t)$, $t \in (0,$ T), айналу, сонымен қатар $u_T\ (t) := u(x,\ T),\ x \in (0,\ \ell)$ соңғы уақытта $T > 0$ ауытқуы өлшенетін шығыстар ретінде берілген деп болжанады. Тихонов функционалдық $J(F) := (1/2)\ ||\ \theta - u_x(0,\cdot)||^2_{L^2(0,T)} + (1/2)||u_T - u(\cdot,T)||^2_{L^2(0,l)}$ Тикхонов функциясы үшін минимизациялау есебі ретінде кері есепті қайта тұжырымдау үшін енгізілген. Бұл функция үшін айқын градиент формуласы алынған. Осы формула негізінде қарастырылатын кері есеп үшін конъюгаттық градиент алгоритмі жасалған. Бұл

алгоритм белгісіз кеңістіктік-уақыттық жүктемені шусыз, сондай-ақ кездейсоқ шулы өлшенген шығыстардан жоғары дәлдікпен қалпына келтіруге мүмкіндік береді.

Түйін сөздер: Эйлер-Бернулли шоғыры, кері коэффициент есебі, Нейманнан Нейманға операторы, квазишешімнің бар болуы, Фреше градиенті.

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Метод реконструкции распределения неизвестных пространственно-временных нагрузок в конструкции на основе вязкоупругости в балочной связи Эйлера-Бернулли

Аннотация. В данной работе изучается новая математическая модель и новый подход к задаче реконструкции неизвестной пространственно-временной нагрузки $F(x,t)$ от обратного источника на просто опертом неоднородном пучке Эйлера-Бернулли, подчиняющемся уравнению $p(x)u_{tt} + \mu(x)u_t + (r(x)u_{xx})_{xx} + k_L u = F(x,t)$, $(x,t) \in (0,\ell) \times (0, T)$, опирающемся на вязкоупругое основание. Вращение $\theta(t) := u_x(0, t)$, $t \in (0, T)$, а также отклонение $u_T(t) := u(x, T)$, $x \in (0, \ell)$ в последний момент времени $T > 0$ при считаются измеряемыми выходами. Функционал Тихонова $J(F) := (1/2)\,||\,\theta - u_x(0,\cdot)||^2_{L^2(0,T)} + (1/2)||u_T - u(\cdot,T)||^2_{L^2(0,l)}$ вводится для переформулировки обратной задачи как задачи минимизации функции Тикимхонова. Для этой функции получена явная формула градиента. На основе этой формулы построен алгоритм сопряженного градиента для рассматриваемой обратной задачи. Этот алгоритм позволяет восстановить неизвестную пространственно-временную нагрузку из высокоточных измеренных выходов без шума, а также со случайным шумом.

Ключевые слова: расслоение Эйлера-Бернулли, обратная коэффициентная задача, оператор Неймана-Неймана, существование квазирешения, градиент Фреше.

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